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Efficient Use of Auxiliary Information in Estimating Finite Population Variance in Sample Surveys

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Abstract

This paper addresses the problem of estimating the finite population variance of the study variable y using information on the known population variance of the auxiliary variable x in sample surveys. We have suggested a class of estimators for population variance using information on population variance of x. The bias and mean squared error of the suggested class of estimators up to first order of approximation was obtained. Preference regions were derived under which the suggested class of estimators is more efficient than the usual unbiased estimator, Das and Tripathi (1980) estimators, Isaki (1983) ratio estimator, Singh et al (1973, 1988) estimator and Gupta and Shabbir (2007) estimator. An empirical study as well as simulation study were carried out in support of the present study.

Key words: Study variable, Auxiliary variable, Class of estimators, Bias, Mean squared error.

1. Introduction

It is tradition to use the auxiliary information at the estimation stage in improving the precision of the estimates of population parameters such as mean and variance. A large amount of work has been carried out towards the estimation of population mean $\bar{Y}$ of the study variable y in the presence of auxiliary information by various authors including Cochran (1940), Robson (1957), Singh, M.P. (1965, 1967), Srivastava (1971, 1980), Srivastava and Jhajj (1980), Sahai and Sahai (1985), Ray and Singh (1981), Gupta (1978), Adhvaryu and Gupta (1983), Singh and Upadhyaya (1986), Singh, H. P. (1986, 1987), Singh and Singh (1984), Tracy et al (1996), Bahl and Tuteja (1991), Singh, S. (2003), Reddy (1978), Walsh (1970), Vos (1980), Singh and Ruiz Espejo (2003), Singh

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and Tailor (2005), Singh et al (2012), Singh and Yadav (2020) and Singh and Nigam (2020) and the references cited therein. However in many situations of practical importance, the problem of estimation of population variance S_y^2 of the study variable y deserves special attention. Singh, Pandey and Hirano (1973) and Searls and Intrapanich (1990) forwarded an improved estimator that utilizes the kurtosis $(\beta_2(y))$ of the study variable y. Later various authors including Das and Tripathi (1978), Isaki (1983), Srivastava and Jhajj (1980), Singh, Upadhyaya and Namjoshi (1988), Gupta and Shabbir (2007), Solanki and Singh (2013), Singh and Solanki (2013a, 2013b), Yadav et al (2013), Pal and Singh (2018), Singh et al. (2003) among others, have paid their attention towards the estimation of population variance S_y^2 of the study variable y using information on population variance S_x^2 of the auxiliary variable x and suggested different estimators for population variance S_y^2. The goal of this paper is to suggest a new class of estimators for population variance S_y^2 utilizing the knowledge on population variance S_x^2 of the auxiliary variable x. The properties of the envisaged class of estimators up to the first order of approximation are studied. The present study is supported through numerical illustration.

2. Notations and Expected Values

Let $U = \{U_1, U_2, \dots, U_N\}$ be a finite population of N units. Let y and x be the study and auxiliary variables respectively. The aim is to estimate the population variance S_y^2 of y using information on population variance S_x^2 of x. A simple random sample (SRS) of size n ($<N$) is drawn from U without replacement (WOR) to estimate S_y^2 of y when S_x^2 of x is known. Let us denote:

$S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})^2$: The population variance/mean square of the study variable y;

$S_x^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(x_i - \bar{X})^2$: The population variance/mean square of x,

$\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$: The population mean of y,

$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$: The population mean of x,

$\beta_2(y) = \frac{\sum_{i=1}^{N}(y_i-\bar{Y})^4}{\left(\sum_{i=1}^{N}(y_i-\bar{Y})^2\right)^2}$: The coefficient of kurtosis of y,

$\beta_2(x) = \frac{\sum_{i=1}^{N}(x_i-\bar{X})^4}{\left(\sum_{i=1}^{N}(x_i-\bar{X})^2\right)^2}$: The coefficient of kurtosis of x,

$$\gamma = \frac{(N-1)\sum_{i=1}^{N}(y_i - \bar{Y})^2(x_i - \bar{X})^2}{(\sum_{i=1}^{N}(y_i - \bar{Y})^2)(\sum_{i=1}^{N}(x_i - \bar{X})^2)}$$

Population size N is large enough so that the finite population correction (fpc) term $\left(1-\frac{n}{N}\right)=(1-f)\cong 1$ is ignored and

$S_y^2=\mu_2(y),\ S_x^2=\mu_2(x), \beta_2(y)=\frac{\mu_4(y)}{\mu_2^2(y)}=\frac{\mu_4(y)}{S_y^4}$,

$\beta_2(x)=\frac{\mu_4(x)}{\mu_2^2(x)}=\frac{\mu_4(x)}{S_x^4}$,

$\gamma=\frac{\mu_{22}(y,x)}{\mu_2(y)\mu_2(x)}=\frac{\mu_{22}(y,x)}{S_y^2S_x^2}, \mu_2(y)=\frac{1}{N}\sum_{i=1}^{N}(y_i-\bar{Y})^2$,

$\mu_2(x)=\frac{1}{N}\sum_{i=1}^{N}(x_i-\bar{X})^2$,

$\mu_{22}(y,x)=\frac{1}{N}\sum_{i=1}^{N}(y_i-\bar{Y})^2\,(x_i-\bar{X})^2$.

For a SRS of size n, we have

$s_y^2=\frac{1}{(n-1)}\sum_{i=1}^{n}(y_i-\bar{y})^2$: sample variance/mean square of y,

$s_x^2=\frac{1}{(n-1)}\sum_{i=1}^{n}(x_i-\bar{x})^2$: sample variance/mean square of x,

$s_y^2=S_y^2(1+e_0), s_x^2=S_x^2(1+e_1)$

such that

$$E(e_0)=E(e_1)=0$$

and to the first degree of approximation and ignoring fpc we have

$E(e_0^2)=\frac{1}{n}(\beta_2(y)-1), E(e_1^2)=\frac{1}{n}(\beta_2(x)-1)$ and $E(e_0e_1)=\frac{1}{n}(\gamma-1)$.

3. Reviewing Some Existing Estimators

The usual unbiased estimator of S_y^2 is given by

$$t_0=s_y^2=\frac{1}{(n-1)}\sum_{i=1}^{n}(y_i-\bar{y})^2. \tag{3.1}$$

The variance /mean square of s_y^2 under SRSWOR scheme (ignoring fpc) is given by

$$MSE(t_0)=\frac{S_y^4}{n}(\beta_2(y)-1). \tag{3.2}$$

Utilizing knowledge on the kurtosis $\beta_2(y)$ of y Singh, Pandey and Hirano (1973) and Searls and Intrapanich (1991) envisaged the following class of estimators for S_y^2 as

$$t_{SPH}=ws_y^2, \tag{3.3}$$

where w is a constant such that the mean squared error (MSE) of t_{SPH} is minimum.

The mean squared error of t_{SPH} ignoring fpc is given by

$$MSE(t_{SPH})=S_y^4\left[1+w^2\left\{1+\frac{\beta_2(y)-1}{n}\right\}-2w\right] \tag{3.4}$$

which is minimum when

$$w=\frac{n}{(n+\beta_2(y)-1)}=w_{(opt)}\text{ (say)}. \tag{3.5}$$

This leads to the resulting estimator

$$t_{SPH} = \frac{n}{(n+\beta_2(y)-1)} s_y^2. \tag{3.6}$$

Substitution of $w_{(opt)}$ at (3.5) in (3.4) yields the MSE of t_{SPH} as

$$MSE(t_{SPH}) = S_y^4 \frac{(\beta_2(y)-1)}{(n+\beta_2(y)-1)}. \tag{3.7}$$

When population variance S_x^2 of x is known, Isaki (1983) suggested a ratio estimator for population variance S_y^2 of y as

$$t_R = s_y^2 \frac{S_x^2}{s_x^2}. \tag{3.8}$$

To the first degree of approximation, the MSE of the ratio estimator t_R ignoring fpc term is given by

$$MSE(t_R) = \frac{S_y^4}{n}[\beta_2(y) + \beta_2(x) - 2\gamma]. \tag{3.9}$$

When S_x^2 is known, Das and Tripathi (1978) suggested the following classes of estimators of S_y^2 as

$$t_{DT1} = s_y^2 \left(\frac{S_x^2}{s_x^2}\right)^\alpha \tag{3.10}$$

and

$$t_{DT2} = s_y^2 \frac{S_x^2}{\{S_x^2+\alpha(s_x^2-S_x^2)\}}, \tag{3.11}$$

where α being suitably chosen constant.

The common minimum MSE of t_{DTi} (i=1,2) to the first degree of approximation, is given by

$$MSE_{\min}(t_{DTi}) = \frac{S_y^4}{n}\left[(\beta_2(y)-1) - \frac{(\gamma-1)^2}{(\beta_2(x)-1)}\right] \tag{3.12}$$

which equals to the minimum MSE of the difference estimator

$$t_D = s_y^2 + d(S_x^2 - s_x^2),$$

where ‘d’ is a suitable chosen constant to be determined such that MSE of t_D is the least.

Singh, Upadhyaya and Namjoshi (1988) proposed a class of difference type estimators for S_y^2 as

$$t_{SUN} = w_1 s_y^2 + w_2(S_x^2 - s_x^2), \tag{3.14}$$

where (w_1, w_2) are suitable chosen constants.

The mean squared error of the estimator t_{SUN} ignoring fpc term is given by

$$MSE(t_{SUN}) = S_y^4[1 + w_1^2 a_1 + w_2^2 a_2 - 2w_1 w_2 a_3 - 2w_1], \tag{3.15}$$

where

$$a_1 = \left[1 + \frac{1}{n}(\beta_2(y)-1)\right], a_2 = \frac{1}{nR^2}(\beta_2(x)-1), a_3 = \frac{1}{nR}(\gamma-1),\ \ R = \frac{S_y^2}{S_x^2}.$$

The MSE(t_{SUN}) at (3.15) is minimum when

$$\left.\begin{aligned} w_1 &= \frac{a_2}{(a_1a_2-a_3^2)} = w_{10}(say) \\ w_2 &= -\frac{a_3}{(a_1a_2-a_3^2)} = w_{20}(say) \end{aligned}\right\}. \tag{3.16}$$

Thus, the resulting minimum MSE of t_{SUN} is given by

$$MSE_{\min}(t_{SUN}) = S_y^4\left[1-\frac{a_2}{(a_1a_2-a_3^2)}\right]. \tag{3.17}$$

Gupta and Shabbir (2007) envisaged the following class of estimators for S_y^2 as

$$t_{GS} = \left[w_1 s_y^2 + w_2(S_x^2 - s_x^2)\right] exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right), \tag{3.18}$$

where (w_1, w_2) are suitable chosen constants.

The MSE of t_{GS} to the first degree of approximation (ignoring fpc term) is given by

$$MSE(t_{GS}) = S_y^4[1 + w_1^2 b_1 + w_2^2 b_2 + 2w_1w_2b_3 - 2w_1b_4 - 2w_2b_5], \tag{3.19}$$

where

$$b_1 = \left[1+\frac{1}{n}(\beta_2(y)+\beta_2(x)-2\gamma)\right], b_2 = \frac{1}{nR^2}(\beta_2(x)-1),$$
$$b_3 = \frac{1}{nR}(\beta_2(x)-\gamma), b_4 = \left[1+\frac{1}{2n}\left\{\frac{3}{4}(\beta_2(x)-1)-\gamma+1\right\}\right],$$
$$b_5 = \frac{1}{2nR}(\beta_2(x)-1).$$

The $MSE(t_{GS})$ at (3.19) is minimum when

$$\left.\begin{aligned} w_1 &= \frac{(b_2b_4-b_3b_5)}{(b_1b_2-b_3^2)} = w_{10(1)}(say) \\ w_2 &= -\frac{(b_1b_5-b_3b_4)}{(b_1b_2-b_3^2)} = w_{20(2)}(say) \end{aligned}\right\}. \tag{3.20}$$

Thus, the least MSE of t_{GS} is given by

$$MSE_{\min}(t_{GS}) = S_y^4\left[1-\frac{(b_2b_4^2-2b_3b_4b_5+b_1b_5^2)}{(b_1b_2-b_3^2)}\right]. \tag{3.21}$$

One may also consider a class of estimators for S_y^2 as

$$t_{SS} = \left[w_1 s_y^2 + w_2(S_x^2 - s_x^2)\right]\left(\frac{S_x^2}{s_x^2}\right). \tag{3.22}$$

The bias and MSE of the estimator t_{SS} to the first degree of approximation (ignoring fpc term) are respectively given by

$$B(t_{SS}) = S_y^2[w_1c_4 + w_2c_5 - 1], \tag{3.23}$$

$$MSE(t_{SS}) = S_y^4[1 + w_1^2c_1 + w_2^2c_2 + 2w_1w_2c_3 - 2w_1c_4 - 2w_2c_5], \tag{3.24}$$

where

$$c_1 = \left[1+\frac{1}{n}(\beta_2(y)+3\beta_2(x)-4\gamma)\right], c_2 = \frac{1}{nR^2}(\beta_2(x)-1),$$
$$c_3 = \frac{1}{nR}(2\beta_2(x)-\gamma+1), c_4 = \left[1+\frac{1}{n}(\beta_2(x)-\gamma)\right], c_5 = \frac{1}{nR}(\beta_2(x)-1),$$

The $MSE(t_{SS})$ at (3.24) is minimum for

$$\left.\begin{aligned} w_1 &= \frac{(c_2c_4-c_3c_5)}{(c_1c_2-c_3^2)} = w_{10(2)}(say) \\ w_2 &= -\frac{(c_1c_5-c_3c_4)}{(c_1c_2-c_3^2)} = w_{20(2)}(say) \end{aligned}\right\}. \tag{3.25}$$

Thus, the resulting minimum MSE of t_{SS} is given by

$$MSE_{\min}(t_{SS}) = S_y^4\left[1-\frac{(c_2c_4^2-2c_3c_4c_5+c_1c_5^2)}{(c_1c_2-c_3^2)}\right]. \tag{3.26}$$

In the following section we have made an effort to develop a new class of estimators for population variance S_y^2 using the knowledge of S_x^2 and its properties are studied. The proposed study is well supported through numerical illustration.

4. The Proposed Class of Estimators

We suggest the following class of estimators for S_y^2 of the study variable y as

$$T = w_1 s_y^2\frac{S_x^2}{[S_x^2+\eta(s_x^2-S_x^2)]} + w_2 s_y^2\, exp\left[\frac{\eta(S_x^2-s_x^2)}{2S_x^2+\eta(s_x^2-S_x^2)}\right]. \tag{4.1}$$

Expressing T in terms e's we have

$$T = S_y^2\left[w_1(1+e_0)(1+\eta e_1)^{-1} + w_2(1+e_0)\, exp\left[\frac{-\eta e_1}{2+\eta e_1}\right]\right]. \tag{4.2}$$

Expanding the right-hand side of (4.2), multiplying out and neglecting terms of e's having power greater than two, we have

$$(T-S_y^2) = S_y^2\left[w_1\{1+e_0-\eta e_1-\eta e_0e_1+\eta^2e_1^2\} + w_2\left\{1+e_0-\frac{\eta e_1}{2}-\frac{\eta e_0 e_1}{2}+\frac{3\eta^2e_1^2}{8}\right\}\right]$$

or

$$(T-S_y^2) = S_y^2\left[w_1(1+e_0-\eta e_1-\eta e_0e_1+\eta^2e_1^2) + w_2\left(1+e_0-\frac{\eta e_1}{2}-\frac{\eta e_0e_1}{2}+\frac{3\eta^2e_1^2}{8}\right)-1\right]. \tag{4.3}$$

Taking expectation of both sides of (4.3) we get the bias of T to the first degree of approximation as

$$B(T) = S_y^2[w_1\Sigma_4 + w_2\Sigma_5 - 1], \tag{4.4}$$

where

$$\Sigma_4 = \left[1+\frac{1}{n}\{\eta^2(\beta_2(x)-1)-\eta(\gamma-1)\}\right],$$
$$\Sigma_5 = \left[1+\frac{1}{n}\left\{\frac{3}{8}\eta^2(\beta_2(x)-1)-\frac{\eta}{2}(\gamma-1)\right\}\right].$$

Squaring both sides of (4.3) and neglecting terms of e's having power greater than two, we have

$$\left(T-S_y^2\right)^2$$
$$=S_y^4\begin{bmatrix}1+w_1^2(1+2e_0-2\eta e_1+e_0^2-4\eta e_0e_1+3\eta^2e_1^2)+w_2^2(1+2e_0-\eta e_1+e_0^2-2\eta e_0e_1+\eta^2e_1^2)\\ +2w_1w_2\left(1+2e_0-\frac{3\eta e_1}{2}+e_0^2-3\eta e_0e_1+\frac{15\eta^2e_1^2}{8}\right)-2w_1(1+e_0-\eta e_1-\eta e_0e_1+\eta^2e_1^2)\\ -2w_2\left(1+e_0-\eta e_1-\eta e_0e_1+\frac{\eta^2e_1^2}{2}\right)\end{bmatrix}.$$

Taking the expectation of both sides of the above expressions we get the MSE of T to the first degree of approximation as

$$MSE(T)=S_y^4[1+w_1^2\Sigma_1+w_2^2\Sigma_2+2w_1w_2\Sigma_3-2w_1\Sigma_4-2w_2\Sigma_5], \tag{4.5}$$

where

$$\Sigma_1=\left[1+\frac{1}{n}\{(\beta_2(y)-1)-4\eta(\gamma-1)+3\eta^2(\beta_2(x)-1)\}\right],$$
$$\Sigma_2=\left[1+\frac{1}{n}\{(\beta_2(y)-1)-2\eta(\gamma-1)+\eta^2(\beta_2(x)-1)\}\right],$$
$$\Sigma_3=\left[1+\frac{1}{n}\left\{(\beta_2(y)-1)-3\eta(\gamma-1)+\frac{15}{8}\eta^2(\beta_2(x)-1)\right\}\right].$$

Differentiating (4.5) with respect w_1 and w_2 and equating them to zero, we have

$$\begin{bmatrix}\Sigma_1\Sigma_3\\ \Sigma_3\Sigma_2\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix}=\begin{bmatrix}\Sigma_4\\ \Sigma_5\end{bmatrix}. \tag{4.6}$$

After simplification of (4.6) we get the optimum values of w_1 and w_2 as

$$\left.\begin{aligned}w_1&=\frac{(\Sigma_2\Sigma_4-\Sigma_3\Sigma_5)}{(\Sigma_1\Sigma_2-\Sigma_3^2)}=w_{1(opt)}\\ w_2&=\frac{(\Sigma_1\Sigma_5-\Sigma_3\Sigma_4)}{(\Sigma_1\Sigma_2-\Sigma_3^2)}=w_{2(opt)}\end{aligned}\right\}. \tag{4.7}$$

Substitution of (4.7) in (4.5) yields the minimum MSE of T as

$$MSE_{\min}(T)=S_y^4\left[1-\frac{(\Sigma_2\Sigma_4^2-2\Sigma_3\Sigma_4\Sigma_5+\Sigma_1\Sigma_5^2)}{(\Sigma_1\Sigma_2-\Sigma_3^2)}\right]. \tag{4.8}$$

which holds true if

$$0<\frac{(\Sigma_2\Sigma_4^2-2\Sigma_3\Sigma_4\Sigma_5+\Sigma_1\Sigma_5^2)}{(\Sigma_1\Sigma_2-\Sigma_3^2)}<1 \text{ and } \left(\Sigma_1\Sigma_2-\Sigma_3^2\right)>0$$

5. Efficiency Comparison

From (3.2) and (3.7) we have

$$MSE\left(s_y^2\right)-MSE(t_{SPH})=\frac{S_y^4\left(\beta_2(y-1)\right)^2}{n(n+\beta_2(y)-1)}\geq 0$$

which gives the inequality

$$MSE(t_{SPH})\leq MSE\left(s_y^2\right). \tag{5.1}$$

This shows that the Singh et al (1973) estimator t_{SUN} is more efficient than usual unbiased estimator s_y^2.

From (3.2), (3.9) and (3.12) we have

$$MSE\left(s_y^2\right) - MSE_{\min}\left(t_j\right) = \frac{S_y^4(\gamma-1)^2}{n(\beta_2(x)-1)} \geq 0, \tag{5.2}$$

$$MSE(t_R) - MSE_{\min}\left(t_j\right) = \frac{S_y^4\left(\beta_2^*(x)-\gamma^*\right)^2}{n(\beta_2(x)-1)} \geq 0, \tag{5.3}$$

where

$$j = \text{DT1, DT2, D};\ \beta_2^*(x) = (\beta_2(x)-1) \text{ and } \gamma^* = (\gamma-1).$$

It follows from (5.2) and (5.3) that the estimators (t_{DT1}, t_{DT2}, t_D) are better than usual unbiased estimator s_y^2 and ratio estimator t_R due to Isaki (1983).

From (3.12) and (3.17) we have

$$MSE_{\min}\left(t_j\right) - MSE_{\min}(t_{SUN}) = \frac{S_y^4\left(a_1a_2-a_2-a_3^2\right)^2}{a_2\left(a_1a_2-a_3^2\right)} \geq 0 \tag{5.4}$$

j= DT1, DT2, D;

It follows from (5.4) that the estimator t_{SUN} is more efficient than $s_y^2, t_R, t_{DT1}, t_{DT2}$ and t_D.

From (3.17) and (4.8) we have that

$$MSE_{\min}(T) < MSE_{\min}(t_{SUN}) \text{ if}$$

$$\frac{\left(\Sigma_2\Sigma_4^2-2\Sigma_3\Sigma_4\Sigma_5+\Sigma_1\Sigma_5^2\right)}{\left(\Sigma_1\Sigma_2-\Sigma_3^2\right)} > \frac{a_2}{\left(a_1a_2-a_3^2\right)}. \tag{5.5}$$

From (3.21) and (4.8) it is observed that

$$MSE_{\min}(T) < MSE_{\min}(t_{GS}) \text{ if}$$

$$\frac{\left(\Sigma_2\Sigma_4^2-2\Sigma_3\Sigma_4\Sigma_5+\Sigma_1\Sigma_5^2\right)}{\left(\Sigma_1\Sigma_2-\Sigma_3^2\right)} > \frac{\left(b_2b_4^2-2b_3b_4b_5+b_1b_5^2\right)}{\left(b_1b_2-b_3^2\right)}. \tag{5.6}$$

From (3.26) and (4.8) we have that

$$MSE_{\min}(T) < MSE_{\min}(t_{SS}) \text{ if}$$

$$\frac{\left(\Sigma_2\Sigma_4^2-2\Sigma_3\Sigma_4\Sigma_5+\Sigma_1\Sigma_5^2\right)}{\left(\Sigma_1\Sigma_2-\Sigma_3^2\right)} > \frac{\left(c_2c_4^2-2c_3c_4c_5+c_1c_5^2\right)}{\left(c_1c_2-c_3^2\right)}. \tag{5.7}$$

Thus, the proposed class of estimators T is more efficient than the estimators t_{SUN}, t_{GS} and t_{SS} as long as the conditions (5.5), (5.6) and (5.7) are satisfied respectively. Hence under the condition (5.5) the proposed class of estimators T is also more efficient than the estimators s_y^2, t_R, t_{DT1}, t_{DT2}, t_D and t_{SPH}.

6. Empirical Study

To illustrate the performance of the suggested class of estimators T over the estimators $s_y^2, t_{SPH}, t_R, t_{DT1}, t_{DT2}, t_D, t_{SUN}, t_{GS}$ and t_{SS}, we consider two natural data sets earlier considered by Das and Tripathi (1982), Kadilar and Cingi (2007) and Singh and Solanki (2013).

Population-I The population consists of 353 villages /towns/ward under Panskura Police Station, (Source: Census 1961, West Bengal, District Census Hand Book, Mindnapore.) The characters y and x are number of persons and area of villages/towns/ward in acres respectively.

For this population, the required parameters were obtained as follows:

S_y^2 = 412624.88, S_x^2 = 40533.195, γ = 12.3063,
$\beta_2(x)$ = 16.3895, $\beta_2(y)$ = 15.05, N = 353, n= 30.

Population-II The data sets earlier used by Kadilar and Cingi (2007) and Singh and Solanki (2013).

In this population data set the level of apple production amount (in 100 tones) is a study variable y and number of apple trees is an auxiliary variable x in 104 villages of the East Anatolia Region of Turkey in 1999. The required values of the parameter are:

S_y^2 = 136.189, S_x^2 = 530202800.90, γ = 14.398,
$\beta_2(y)$ = 16.523, $\beta_2(x)$ = 17.516, N = 104, n = 20.

We have computed the percent relative efficiencies (PRE's) of the estimators $s_y^2, t_{SPH}, t_R, t_{DT1}, t_{DT2}, t_D, t_{SUN}, t_{GS}$ and t_{SS} for both population data sets (I and II) and the resulting values are compiled in Tables 6.1 and 6.2 respectively.

Table 6.1. PRE's of different Estimators of Population Variance with respect to usual unbiased estimator s_y^2 for Population-I and Population-II

Estimator	Population-I	Population-II
	PRE (. , s_y^2)	PRE (. , s_y^2)
s_y^2 Usual unbiased estimator	100.00	100.00
t_R Isaki (1983) ratio estimator	205.80	296.07
t_{SPH} Singh, Pandey and Hirano (1973) estimator	146.83	177.62
(t_{DT1}, t_{DT2}, t_D) Das and Tripathi (1978) estimator	244.62	333.52
t_{SUN} Singh, Upaadhyaya and Namjoshi (1988) estimator	291.45	411.13
t_{GS} Gupta and Shabbir (2007) estimator	**342.77**	549.81
t_{SS}	340.78	**779.07**

Table 6.2. PRE's of proposed class of estimators T with respect to usual unbiased estimator s_y^2 for populations I and II

Population I		Population II	
Values of constant η	PRE $\left(T, s_y^2\right)$	Values of constant η	PRE $\left(T, s_y^2\right)$
-5.25	1220.64	-5.25	817.99
-5.00	1289.65	-5.00	864.91
-4.75	1364.36	-4.75	915.85
-4.50	1445.38	-4.50	971.29
-4.25	1533.36	-4.25	1031.73
-4.00	1629.09	-4.00	1097.79
-3.75	1733.39	-3.75	1170.15
-3.50	1847.25	-3.50	1249.60
-3.25	1971.70	-3.25	1337.07
-3.00	2107.95	-3.00	1433.61
-2.75	2257.28	-2.75	1540.44
-2.50	2421.11	-2.50	1659.00
-2.25	2600.93	-2.25	1790.94
-2.00	2798.19	-2.00	1938.17
-1.75	3014.12	-1.75	2102.89
-1.50	3249.25	-1.50	2287.53
-1.25	3502.13	-1.25	2494.66
-1.00	3765.87	-1.00	2726.41
-0.75	4015.81	-0.75	2982.79
-0.50	4147.05	-0.50	3255.02
-0.25	3161.99	-0.25	3492.46
0.25	7368.44	0.25	8020.37
0.50	7891.91	0.50	6778.99
0.75	8839.35	0.75	7851.08
1.00	10358.80	1.00	10307.08
1.25	13373.12	1.25	19118.06
1.50	25878.68	**1.42**	**895292.40**
1.63	**401889.70**	1.90	860.29
2.00	1045.41	2.00	1942.39
2.25	4238.41	2.25	3689.14
2.50	5703.97	2.50	4797.45
2.75	6537.20	2.75	5576.82
3.00	7023.40	3.00	6123.91
3.25	7271.59	3.25	6473.92
3.50	7338.32	3.50	6645.46
3.75	7262.57	3.75	6657.09
4.00	7076.72	4.00	6532.59
4.25	6809.62	4.25	6300.44
4.50	6486.87	4.50	5990.92
4.75	6130.28	4.75	5632.61

Table 6.2. PRE's of proposed class of estimators T with respect to usual unbiased estimator s_y^2 for populations I and II (cont.)

Population I		Population II	
Values of constant η	PRE (T, s_y^2)	Values of constant η	PRE (T, s_y^2)
5.00	5757.64	5.00	5249.97
5.25	5382.63	5.25	4862.17
5.50	5015.27	5.50	4482.99
5.75	4662.38	5.75	4121.53
6.00	4328.29	6.00	3783.14
6.25	4015.37	6.25	3470.41
6.50	3724.59	6.50	3183.99
6.75	3455.92	6.75	2923.29
7.00	3208.69	7.00	2686.98
7.25	2981.82	7.25	2473.30
7.50	2774.01	7.50	2280.33
7.75	2583.84	7.75	2106.14
8.00	2409.89	8.00	1948.86
8.25	2250.75	8.25	1806.73
8.50	2105.11	8.50	1678.15
8.75	1971.72	8.75	1561.66
9.00	1849.44	9.00	1455.97
9.25	1737.22	9.25	1359.90

7. Simulation Study

To access the performance of the proposed class of estimators a simulation study is performed using R-software to verify the theoretical results. We have generated artificial population of two variables (y, x) based on regression model as x = rnorm $(N, 0, 1)$ and y = x + rnorm $(N, 0, 1)$ of size N. We have generated two populations:

Population-I: $N = 5000$, $n = 2000$; Population-II: $N = 10000$, $n = 4000$.

Table 7.1. PRE's of different Estimators of Population Variance with respect to usual unbiased estimator s_y^2 for simulated Populations I and II

Estimator	Population-I	Population-II
	PRE $(. , s_y^2)$	PRE $(. , s_y^2)$
s_y^2 Usual unbiased estimator	100.00	100.00
t_R Isaki (1983) estimator	25.56	25.92
t_{SPH} Singh, Pandey and Hirano (1973) estimator	99.95	99.98
(t_{DT1}, t_{DT2}, t_D) Das and Tripathi (1978) estimator	1173.71	729.83
t_{SUN} Singh, Upaadhyaya and Namjoshi (1988) estimator	1173.65	729.81
t_{GS}Gupta and Shabbir (2007) estimator	1173.45	729.74
t_{SS}	1173.66	729.81

Table 7.2. PRE's of proposed class of estimators T with respect to usual unbiased estimator s_y^2 for simulated populations I and II

Population I		Population II	
Values of constant η	PRE (T, s_y^2)	Values of constant η	PRE (T, s_y^2)
-13.00	480401.38	-13.00	1257339.40
-12.75	516601.08	-12.75	1325095.50
-12.50	555811.16	-12.50	1395274.40
-12.25	598235.74	-12.25	1467685.90
-12.00	644072.75	-12.00	1542089.00
-11.75	693506.78	-11.75	1618190.70
-11.50	746700.10	-11.50	1695645.50
-11.25	803781.66	-11.25	1774057.60
-9.75	1227383.64	-9.75	2237550.70
-9.50	1309460.73	-9.50	2308716.00
-9.25	1393511.34	-9.25	2376775.80
-9.00	1478746.04	-9.00	2441295.80
-8.75	1564256.06	-8.75	2501903.10
-8.50	1649039.04	-8.50	2558296.40
-8.25	1732035.48	-8.25	2610252.30
-8.00	1812174.46	-8.00	2657630.10
-7.75	1888425.13	-7.75	2700371.20
-7.50	1959849.53	-7.50	2738497.30
-7.25	2025651.43	-7.25	2772104.40
-7.00	2085216.23	-7.00	2801355.40
-6.75	2138137.75	-6.75	2826470.40
-6.50	2184229.77	-6.50	2847716.30
-6.25	2223521.84	-6.25	2865396.10
-6.00	2256241.20	-6.00	2879837.80
-5.75	2282783.69	-5.75	2891384.40
-5.50	2303677.91	-5.50	2900384.40
-5.25	2319546.34	-5.25	2907183.70
-5.00	2331067.19	-5.00	2912118.40
-4.75	2338939.59	-4.75	2915509.00
-4.50	2343853.86	-4.50	2917656.00
-4.25	2346467.62	-4.25	2918835.70
-4.00	2347387.74	-4.00	2919298.30
-3.75	2347157.61	-3.75	2919265.50
-3.50	2346248.83	-3.50	2918929.40
-3.25	2345056.27	-3.25	2918451.90
-3.00	2343895.73	-3.00	2917963.90
-2.75	2343002.95	-2.75	2917565.40
-2.50	2342533.61	-2.50	2917325.30
-2.25	2342563.39	-2.25	2917281.50
-2.00	2343088.09	-2.00	2917440.70

Table 7.2. PRE's of proposed class of estimators T with respect to usual unbiased estimator s_y^2 for simulated populations I and II (cont.)

Population I		Population II	
Values of constant η	PRE (T, s_y^2)	Values of constant η	PRE (T, s_y^2)
-1.75	2344023.33	-1.75	2917778.50
-1.50	2345203.99	-1.50	2918239.30
-1.25	2346383.36	-1.25	2918736.10
-1.00	2347233.10	-1.00	2919150.50
-0.75	2347344.01	-0.75	2919333.60
-0.50	2346226.93	-0.50	2919102.60
-0.25	2343320.54	-0.25	2918247.60
0.25	2329569.50	0.25	2913680.20
0.50	2317332.74	0.50	2909398.80
0.75	2300564.35	0.75	2903373.30
1.00	2278568.07	1.00	2895266.80
1.25	2250709.14	1.25	2884729.40
1.50	2216455.49	1.50	2871407.00
1.75	2175418.12	1.75	2854945.30
2.00	2127388.91	2.00	2835001.80
2.25	2072370.42	2.25	2811254.50
2.50	2010593.42	2.50	2783413.30
2.75	1942519.02	2.75	2751231.80
3.00	1868823.93	3.00	2714517.50
3.25	1790369.45	3.25	2673143.00
3.50	1708156.83	3.50	2627054.60
3.75	1623273.93	3.75	2576279.00
4.00	1536838.68	4.00	2520927.40
4.25	1449945.03	4.25	2461196.60
4.50	1363616.11	4.50	2397366.20
4.75	1278767.89	4.75	2329792.60
5.00	1196184.82	5.00	2258900.10
5.25	1116507.36	5.25	2185168.50
5.50	1040230.08	5.50	2109119.50
5.75	967708.11	5.75	2031301.30
6.00	899169.65	6.00	1952273.20
6.25	834732.14	6.25	1872590.10
6.50	774419.98	6.50	1792789.50
6.75	718182.47	6.75	1713378.80
7.00	665910.63	7.00	1634826.00
7.25	617452.29	7.25	1557552.20
7.50	572625.18	7.50	1481927.10
7.75	531227.79	7.75	1408266.40
8.00	493048.19	8.00	1336831.50
8.25	457870.94	8.25	1267830.80

Table 7.2. PRE's of proposed class of estimators T with respect to usual unbiased estimator s_y^2 for simulated populations I and II (cont.)

Population I		Population II	
Values of constant η	PRE (T, s_y^2)	Values of constant η	PRE (T, s_y^2)
8.50	425482.40	8.50	1201422.50
8.75	395674.52	8.75	1137718.70
9.00	368247.63	9.00	1076789.40
9.25	343012.21	9.25	1018667.60
9.50	319789.97	9.50	963354.50
9.75	298414.38	9.75	910823.80
10.00	278730.81	10.00	861027.00
10.25	260596.31	10.25	813897.40
10.50	243879.25	10.50	769353.80
10.75	228458.78	10.75	727304.40
11.00	214224.20	11.00	687649.70
11.25	201074.31	11.25	650284.90
11.50	188916.74	11.50	615102.20
11.75	177667.26	11.75	581992.80
12.00	167249.13	12.00	550848.10
12.25	157592.45	12.25	521561.10
12.50	148633.62	12.50	494027.30
12.75	140314.73	12.75	468145.30
13.00	132583.09	13.00	443817.30
13.25	125390.71	13.25	420949.70
13.50	118693.92	13.50	399453.10
13.75	112452.92	13.75	379242.50
14.00	106631.46	14.00	360237.50
14.25	101196.47	14.25	342362.00
14.50	96117.81	14.50	325544.20
14.75	91367.94	14.75	309716.70
15.00	86921.71	15.00	294816.20
15.25	82756.16	15.25	280783.20
15.50	78850.24	15.50	267562.00
15.75	75184.72	15.75	255100.70
16.00	71741.98	16.00	243350.40
16.25	68505.84	16.25	232265.90
16.50	65461.49	16.50	221804.50
16.75	62595.29	16.75	211926.70
17.00	59894.75	17.00	202595.50
17.25	57348.36	17.25	193776.40
17.50	54945.51	17.50	185437.20
17.75	52676.46	17.75	177548.00
18.00	50532.20	18.00	170080.90

8. Discussion

It is observed from Table 6.1 that in population-I, the estimator t_{GS} due to Gupta and Shabbir (2007) appears to be the best (in the sense of having least MSE) followed by the estimator t_{SS}while in population-II, the estimator $t_{SS}$is the best followed by the estimator t_{GS} due to Gupta and Shabbir (2007).

Comparing the results of Tables 6.1 and 6.2 it is observed that the PRE (T, s_y^2) = **401889.70%** is the largest at η = **1.63**, which is very high as compared to the Gupta and Shabbir (2007) estimator t_{GS} $[PRE(t_{GS}, s_y^2) = 342.77\%]$ in population-I. The PRE (T, s_y^2) is very high as compared to all estimators including t_{GS} in population-I for other values of constant η also. It is further observed from Table 6.2 that in population-II, the maximum PRE (T, s_y^2) = **895292.40%** at η = **1.42**, which is very large as compared to the estimator t_{SS} $[PRE(t_{SS}, s_y^2) = 779.07\%]$. However, for other values of η in population-II, the PRE (T, s_y^2) gives the larger values than the estimator t_{SS}. Thus, from Tables 6.1 and 6.2 it is observed that there is enough scope of selecting the values of η in obtaining estimators better than the estimators $s_y^2, t_{SPH}, t_R, t_{DT1}, t_{DT2}, t_D, t_{SUN}, t_{GS}$ and t_{SS}closed in Table 6.1. Finally, we conclude that the proposed class of estimators perform well as compared to the existing estimators discussed here. So, we recommend the proposed estimator T for its use in practice.

The results of simulation experiments which reveal the ascendance of PRE of the estimators
$s_y^2, t_{SPH}, t_R, t_{DT1}, t_{DT2}, t_D, t_{SUN}, t_{GS}$ and t_{SS} and the proposed class of estimators T with respect to conventional unbiased estimator s_y^2 are displayed in Tables 7.1 and 7.2 for various values of scalar 'η'.

Table 7.1 exhibits that the common PRE due to Das and Tripathi's (1978) estimators t_{DT1}, t_{DT2} and t_D is the largest among the estimators $(s_y^2, t_{SPH}, t_R, t_{SUN}, t_{GS,} t_{SS,} t_{DT1}, t_{DT2}, t_D)$ followed by the estimators t_{SS}. The PREs of the estimators $t_{DT1}, t_{DT2}, t_D, t_{SUN}, t_{GS}$ and t_{SS} with respect to s_y^2 are almost same in both the population I and II. It follows that the performance of the estimators $t_{DT1}, t_{DT2}, t_D, t_{SUN}, t_{GS}$ and t_{SS} are almost same. It is further observed that for both the artificial populations I and II, the performance of Isaki (1983) ratio-type estimator t_R and Singh, Pandey and Hirano (1973) estimator t_{SPH} even worse than the usual unbiased estimator s_y^2 (which does not utilize auxiliary information).

From the perusal of the simulated results summarized in Tables 7.1 and 7.2 for artificial populations I and II, it can be seen that the performance of the suggested class of estimators T is better than the usual unbiased estimator s_y^2, Isaki's (1983) ratio-type estimator t_R, Singh, Pandey and Hirano (1973) estimator t_{SPH}, Das and Tripathi's (1978) estimators (t_{DT1}, t_{DT2}, t_D)Singh, Upadhyaya and Namjoshi (1988) estimator t_{SUN}, Gupta and Shabbir's (2007) estimator t_{GS} and the estimator t_{SS} for various values of the scalar 'η'. Thus, the suggested class of estimators T is recommended for its use in practice based on the simulation study results too.

9. Conclusion

This article addresses the problem of estimating the population variance S_y^2 of a study variable y when information on population variance S_x^2 of the auxiliary variable x is available under simple random sampling without replacement (SRSWOR). We have suggested a class of estimators for population variance S_y^2 of the study variable y using information on population variance S_x^2 of the auxiliary variable x. We have obtained the bias and mean squared error of the suggested class of estimators up to first order of approximation. The optimum conditions are obtained under which the proposed class of estimators has least MSE. The merits of the suggested class of estimators are judged through two natural population data sets. It has been shown empirically that the suggested class of estimators is more efficient than the existing estimators considered here with substantial gain in efficiency. This fact can be seen from Tables 6.1 and 6.2. We have also carried out simulation study based on two artificial populations I and II. We have computed PRE's of different estimators of population variance S_y^2 relative to s_y^2 and the results are presented in Tables 7.1 and 7.2. Larger gain efficiency is observed by using the suggested class of estimators T over other existing estimators for a wide range of scalar "η". Finally, the results theoretically and empirically are very encouraging and useful to the researcher engaged in this area of interest. So, we recommend the proposed estimator for its use in practice.

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